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Three Types Dual Model for Minimax Fractional Programming

J. C. LIU
Section of Mathematics, National Overseas Chinese Student University
P.O. Box 1-1337 Linkou, 24499, Taiwan

Y. KIMURA
Department of Mathematics and Information Science
Graduate School of Science and Technology
Niigata University 950-21, Niigata, Japan

K. TANAKA
Department of Mathematics, Niigata University
950-21, Niigata, Japan

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Abstract—We establish necessary and sufficient optimality conditions for minimax fractional programming involving nonsmooth generalized $(\Im, \rho, \theta)$- convex functions. Three types dual model of [1] are considered and certain duality results have been derived in the framework of generalized $(\Im, \rho, \theta)$-convex functions.

Keywords—Minimax fractional programming, Generalized $(\Im, \rho, \theta)$-convex functions, Duality.

1. INTRODUCTION

In this paper, we consider the following minimax fractional programming problem:

$$v^* = \min_{x \in S} \max_{1 \le i \le p} \left[\frac{f_i(x)}{g_i(x)} \right], \tag{P}$$

where

(A1) $S = \{x \in \mathbb{R}^n;\ h_k(x) \le 0,\ k = 1, 2, \dots, m\}$ is nonempty and compact;
(A2) $f_i : X_0 \mapsto \mathbb{R}$, $g_i : X_0 \mapsto \mathbb{R}$, $i = 1, 2, \dots, p$, and $h_k : X_0 \mapsto \mathbb{R}$, $k = 1, 2, \dots, m$ are locally Lipschitz continuous and X_0 is the open subset of $\mathbb{R}^n$;
(A3) $g_i(x) > 0$, $i = 1, 2, \dots, p$, $x \in S$;
(A4) if g_i is not affine, then $f_i(x) \ge 0$ for all i and all $x \in S$.

Many papers have been devoted to the minimax fractional programming problem in recent decades; see for example [1–21]. In [7], Crouzeix *et al.* have shown that the minimax fractional program can be derived by solving the following minimax nonlinear (nondifferentiable) parametric program:

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$$\min_{x \in S} \max_{1 \le i \le p} \left(f_i(x) - v g_i(x)\right), \tag{P_v}$$

where $v \in \mathbb{R}_+ \equiv [0, \infty)$ is a parameter.

It is clear that (P_v) is equivalent to the following problem (EP_v) for a given v:

$$\begin{aligned} &\min q, \\ &\text{subject to} \quad f_i(x) - v g_i(x) \le q, \qquad i = 1, 2, \dots, p, \\ &\phantom{\text{subject to}} \quad h_k(x) \le 0, \qquad\qquad\quad k = 1, 2, \dots, m. \end{aligned} \tag{EP_v}$$

In [3], Bector *et al.* employed the problem (EP_v) to prove necessary and sufficient optimality conditions for problem (P) and establish various duality results for problem (EP_v) involving differentiable generalized convex functions (or generalized invex functions). Liu [10–12] also adapted the same approach to obtain necessary and sufficient optimality conditions; and he derived duality theorems for generalized fractional programming problems involving either nonsmooth pseudoinvex functions [10] or nonsmooth (F, ρ)-convex functions [11], and duality theorems for generalized fractional variational problems involving generalized (F, ρ)-convex functions [12].

However, all of the above results need constraint qualifications and a constraint qualification that is imposed on the constrains of (P) may not hold for (EP_v) but hold for (P_v). Actually, Ben-Israel *et al.* [22] have given necessary and sufficient conditions for optimality for convex (scalar) programming problems without any condition of a constraint qualification. Recently, Lai *et al.* [1] used the optimality conditions of [22] and problem (P_v) to establish necessary and sufficient optimality conditions for minimax (convex) fractional programming without constraint qualifications and they constructed one parametric and two parametric-free dual models.

In this paper, we also want to use problem (P_v) to establish both parametric and nonparameter necessary and sufficient optimality conditions of (P) involving nonsmooth $(\Im, \rho, \theta)$-convex functions, and use these optimality conditions to consider three types dual model of [1] and establish some duality results for (P). We organize this paper as follows. Some definitions and notations are given in Section 2. In Section 3, we establish necessary and sufficient optimality conditions for (P) involving generalized nonsmooth $(\Im, \rho, \theta)$-convex functions. Finally, duality theorems are presented in Sections 4–6.

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its nonnegative orthant. Let X_0 be an open subset of $\mathbb{R}^n$.

DEFINITION 2.1. *The function $\Gamma : X_0 \mapsto \mathbb{R}$ is said to be Lipschitz on X_0 if there exists $c > 0$ such that for all $y, x \in X_0$,*

$$|\Gamma(y) - \Gamma(x)| \le c\|y - x\|,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$.

For each d in $\mathbb{R}^n$, $\Gamma^\circ(x; d)$ is the *generalized directional derivative of Clarke* [23] defined by

$$\Gamma^\circ(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{[\Gamma(y + td) - \Gamma(y)]}{t}.$$

It then follows that

$$\Gamma^\circ(x; d) = \max\left\{\xi^T d \mid \xi \in \partial\Gamma(x)\right\}, \qquad \text{for any } x \text{ and } d,$$

where $\partial\Gamma(\cdot)$ denotes the *Clarke's generalized gradient* [23].

It is well known that the problem (P) is equivalent (see [6,7]) to the following nonfractional parametric problem:

$$(\mathrm{P}_v) \qquad \min_{x \in S} \max_{1 \le i \le p} \left(f_i(x) - v g_i(x)\right),$$

where $v \in \mathbb{R}_+ \equiv [0, \infty)$ is a parameter. We need the following lemmas.

LEMMA 2.1. *(See [18, Lemma 3.1].) Let v^* be the optimal value of (P), and let $V(v)$ be the optimal value of (P_v) for any fixed $v \in \mathbb{R}_+$ such that (P_v) has an optimal solution. Then x^* is an optimal solution of (P) if and only if x^* is an optimal solution of (P_{v^*}) with optimal value $V(v^*) = 0$.*

LEMMA 2.2. *(See [23, Proposition 2.3.12].) Let $f_1, \dots, f_p$ be Lipschitz functions at x^* and $\alpha_i \in \mathbb{R}$ for all $i = 1, \dots, p$. Then*

(1) $\partial(\sum_{i=1}^p \alpha_i f_i)(x^*) \subset \sum_{i=1}^p \alpha_i \partial f_i(x^*)$,

(2) $\partial[\max_{1\le i\le p} f_i](x^*) \subset \cup\{\sum_{l\in L} \alpha_l \partial f_l(x^*);\ \alpha_l \ge 0,\ \sum_{l\in L} \alpha_l = 1\}$ *where L is the set of indices l for which*
$$f_l(x^*) = \max_{1\le i\le p} f_i(x^*).$$

LEMMA 2.3. *(See [18, Lemma 3.2].) For each $x \in S$, one has*
$$\phi(x) \equiv \max_{1\le i\le p}\left(\frac{f_i(x)}{g_i(x)}\right) = \max_{\beta\in U}\left(\frac{\sum\limits_{i=1}^p \beta_i f_i(x)}{\sum\limits_{i=1}^p \beta_i g_i(x)}\right),$$
where $U = \{\beta \in \mathbb{R}_+^p \mid \sum_{i=1}^p \beta_i = 1\}$.

For convenience, we give the scalar minimization problem as follows:
$$\begin{aligned} &\text{minimize} && N(x), \\ &\text{subject to} && h_k(x) \le 0, \qquad k = 1, 2, \dots, m, \end{aligned} \tag{SP}$$
where $N, h_k : X_0 \mapsto \mathbb{R}$, $k = 1, 2, \dots, m$, are Lipschitz on X_0. We need the following lemma.

LEMMA 2.4. *(See [24, Theorem 6].) If $x^* \in X_0$ is a local minimum for (SP) and a constraint qualification [24] is satisfied, then there exist $z^* = (z_1^*, \dots, z_m^*) \in \mathbb{R}_+^m$ such that*
$$\begin{aligned} 0 &\in \partial N(x^*) + \sum_{k=1}^m z_k^* \partial h_k(x^*), \\ z_k^* h_k(x^*) &= 0, \qquad \text{for all } k = 1, 2, \dots, m. \end{aligned}$$

For simplicity, throughout the paper we denote
$$\begin{aligned} U &= \left\{\alpha \in \mathbb{R}_+^p \;\middle|\; \sum_{i=1}^p \alpha_i = 1\right\}, \\ F(x) &= (f_1(x), \dots, f_p(x)), \\ G(x) &= (g_1(x), \dots, g_p(x)), \qquad \text{and} \\ H(x) &= (h_1(x), \dots, h_m(x)). \end{aligned}$$
For $z \in \mathbb{R}^m$, $z^\top H(x^*) = \sum_{k=1}^m z_k h_k(x^*)$, and $\partial(z^\top H)(x^*) = \sum_{k=1}^m z_k \partial h_k(x^*)$.

We shall use Lemmas 2.1–2.4 to establish the following necessary and sufficient optimality conditions for (P).

THEOREM 2.1. NECESSARY OPTIMALITY CONDITIONS. *Let $x^* \in S$. If x^* is an optimal solution of (P) that the constraint of (P) satisfy Slater's constraint qualification [24]. Then there exist $v^* = \phi(x^*) \in \mathbb{R}_+$, $y^* \in U$, $z^* \in \mathbb{R}_+^m$ such that*
$$0 \in \partial\left(y^{*\top} F\right)(x^*) - v^* \partial\left(y^{*\top} G\right)(x^*) + \partial\left(z^{*\top} H\right)(x^*), \tag{2.1}$$
$$y^{*\top} F(x^*) - v^* y^{*\top} G(x^*) = 0, \tag{2.2}$$
$$z^{*\top} H(x^*) = 0. \tag{2.3}$$

PROOF. If x^* is an optimal solution of (P), by Lemma 2.1, it is an optimal solution of (P_{v^*}) with optimal value $v^* = \max_{1\le i\le p}[f_i(x^*)/g_i(x^*)] \equiv \phi(x^*)$. Thus, by Lemma 2.4, there exist $z^* \in \mathbb{R}^m_+$, such that

$$0 \in \partial\left(\max_{1\le i\le p}(f_i - v^* g_i)\right)(x^*) + \partial\left(z^{*\top}H\right)(x^*)$$

and

$$z^{*\top}H(x^*) = 0.$$

It follows from Lemma 2.2 that there exist $\alpha_l \ge 0$, $l \in L$, $\sum_{l\in L}\alpha_l = 1$, such that

$$0 \in \sum_{l\in L}\alpha_l\left(\partial f_l(x^*) + v^*\partial(-g_l(x^*))\right) + \partial\left(z^{*\top}H\right)(x^*).$$

It is obvious that $v^* = \max_{1\le i\le p}[f_i(x^*)/g_i(x^*)]$ if and only if $\max_{1\le i\le p}[f_i(x^*) - v^* g_i(x^*)] = 0$. Thus, if we set $y_i^* = \alpha_i$ for $i \in L$, as well as, $y_i^* = 0$ for $i \in \{1,2,\dots,p\}\setminus L$, the expressions (2.1)–(2.3) hold. ∎

In order to construct parameter-free duality models for problem (P), we replaced the parameter v^* by $y^{*\top}F(x^*)/y^{*\top}G(x^*)$, and derived another parameter-free versions for necessary conditions as follows.

THEOREM 2.2. *Let $x^* \in S$. If x^* is an optimal solution of (P) and the constraint of (P) satisfy Slater's constraint qualification [24], then there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$ such that*

$$0 \in y^{*\top}G(x^*)\left(\partial\left(y^{*\top}F\right)(x^*) + \partial\left(z^{*\top}H\right)(x^*)\right) - y^{*\top}F(x^*)\,\partial\left(y^{*\top}G\right)(x^*), \tag{2.4}$$

$$z^{*\top}H(x^*) = 0, \tag{2.5}$$

and obtain the optimal value by

$$\phi(x^*) = \frac{y^{*\top}F(x^*)}{y^{*\top}G(x^*)} = \max_{1\le i\le p}\left(\frac{f_i(x^*)}{g_i(x^*)}\right). \tag{2.6}$$

In order to relax the convexity assumption in sufficient optimality conditions for (P), we impose the following nonsmooth $(\Im,\rho,\theta)$-convex functions.

DEFINITION 2.2. *A functional $\Im : X \times X \times \mathbb{R}^n \mapsto \mathbb{R}$ (where $X \subseteq \mathbb{R}^n$) is sublinear if for any $x, x^0 \in X$,*

$$\Im\left(x,x^0;a_1+a_2\right) \le \Im\left(x,x^0;a_1\right) + \Im\left(x,x^0;a_2\right), \qquad \text{for any } a_1,a_2 \in \mathbb{R}^n \tag{2.7}$$

and

$$\Im\left(x,x^0;\alpha a\right) \le \alpha\Im\left(x,x^0;a\right), \qquad \text{for any } \alpha \in \mathbb{R},\ \alpha \ge 0, \text{ and } a \in \mathbb{R}^n. \tag{2.8}$$

From (2.7) and (2.8), it follows $\Im(x,x^0;0) = 0$.

In the above definition, it is obvious that sublinearity is with respect to the third variable. Let us consider a sublinear functional $\Im$ and the function $\Gamma : X \mapsto \mathbb{R}$ (where $X \subseteq \mathbb{R}^n$). We suppose Γ is Lipschitz on X. Let $\rho \in \mathbb{R}$ and $\theta : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}_+$ be such that $\theta(x_1,x_2) \ne 0$ for $x_1 \ne x_2$ in $\mathbb{R}^n$.

DEFINITION 2.3.

(a) *The function Γ is said to be $(\Im,\rho,\theta)$-convex at x^0 if for all $x \in X$, we have*

$$\Gamma(x) - \Gamma\left(x^0\right) \ge \Im\left(x,x^0;\xi\right) + \rho\theta\left(x,x^0\right), \qquad \text{for every } \xi \in \partial\Gamma\left(x^0\right).$$

This function Γ is said to be strongly $\Im$-convex, $\Im$-convex, or weakly $\Im$-convex at x^0, according as $\rho > 0$, $\rho = 0$, or $\rho < 0$.

(b) *The function Γ is said to be $(\Im, \rho, \theta)$-quasiconvex at x^0 if for all $x \in X$ such that $\Gamma(x) \leq \Gamma(x^0)$, we have*

$$\Im\left(x, x^0; \xi\right) \leq -\rho\theta\left(x, x^0\right), \qquad \text{for every } \xi \in \partial\Gamma\left(x^0\right).$$

We say that Γ is strongly $\Im$-quasiconvex, $\Im$-quasiconvex, or weakly $\Im$-quasiconvex at x^0, according as $\rho > 0$, $\rho = 0$, or $\rho < 0$.

(c) *The function Γ is said to be Ponstein [25] $(\Im, \rho, \theta)$-quasiconvex at x^0 if for all $x \in X$ such that $\Gamma(x) < \Gamma(x^0)$, we have*

$$\Im\left(x, x^0; \xi\right) \leq -\rho\theta\left(x, x^0\right), \qquad \text{for every } \xi \in \partial\Gamma\left(x^0\right).$$

This function Γ is said to be Ponstein strongly $\Im$-quasiconvex, Ponstein $\Im$-quasiconvex, or weakly Ponstein $\Im$-quasiconvex at x^0, according as $\rho > 0$, $\rho = 0$, or $\rho < 0$.

(d) *The function Γ is said to be $(\Im, \rho, \theta)$-pseudoconvex at x^0 if for all $x \in X$ such that $\Im(x, x^0; \xi) \geq -\rho\theta(x, x^0)$ for every $\xi \in \partial\Gamma(x^0)$, it results $\Gamma(x) \geq \Gamma(x^0)$.*

We say that Γ is strongly $\Im$-pseudoconvex, $\Im$-pseudoconvex, or weakly $\Im$-pseudoconvex at x^0, according as $\rho > 0$, $\rho = 0$, or $\rho < 0$.

(e) *The function Γ is said to be strictly $(\Im, \rho, \theta)$-pseudoconvex at x^0 if for all $x \in X$, $x \neq x^0$ such that $\Im(x, x^0; \xi) \geq -\rho\theta(x, x^0)$ for every $\xi \in \partial\Gamma(x^0)$, it results $\Gamma(x) > \Gamma(x^0)$.*

3. SUFFICIENT OPTIMALITY CONDITIONS

In this section, we derive some sufficient optimality conditions for (P) involving generalized nonsmooth $(\Im, \rho, \theta)$-convex functions as follows.

THEOREM 3.1. SUFFICIENT OPTIMALITY CONDITIONS. *Let $x^* \in S$, and assume that there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$, such that the conditions (2.4)–(2.6) hold. Let*

$$A(x) = y^{*\top} G\left(x^*\right) y^{*\top} F(x) - y^{*\top} F\left(x^*\right) y^{*\top} G(x),$$
$$B(x) = z^{*\top} H(x), \qquad \text{and} \qquad C(x) = A(x) + y^{*\top} G\left(x^*\right) B(x).$$

If any one of the following conditions holds, then x^* is an optimal solution of (P).

(a) *$y^{*\top} F$ is $(\Im, \rho_1, \theta)$-convex at x^*, $-y^{*\top} G$ is $(\Im, \rho_2, \theta)$-convex at x^*, $z^{*\top} H$ is $(\Im, \rho_3, \theta)$-convex at x^*, and $y^{*\top} G(x^*)\rho_1 + y^{*\top} F(x^*)\rho_2 + y^{*\top} G(x^*)\rho_3 \geq 0$.*

(b) *A is $(\Im,\ \rho_1,\ \theta)$-pseudoconvex at x^*, B is $(\Im,\ \rho_2,\ \theta)$-quasiconvex at x^*, and $\rho_1 + y^{*\top} G(x^*)\rho_2 \geq 0$.*

(c) *A is $(\Im, \rho_1, \theta)$-quasiconvex at x^*, B is strictly $(\Im, \rho_2, \theta)$-pseudoconvex at x^*, and $\rho_1 + y^{*\top} G(x^*)\rho_2 \geq 0$.*

(d) *A is Ponstein $(\Im, \rho_1, \theta)$-quasiconvex at x^*, B is $(\Im, \rho_2, \theta)$-quasiconvex at x^*, and $\rho_1 + y^{*\top} G(x^*)\rho_2 > 0$.*

(e) *C is $(\Im, \rho, \theta)$-pseudoconvex at x^* and $\rho \geq 0$.*

(f) *C is Ponstein $(\Im, \rho, \theta)$-quasiconvex at x^* and $\rho > 0$.*

PROOF. Suppose contrary that x^* were not an optimal solution of (P). Then there exists a feasible solution $x_1 \in S$ such that

$$\phi(x^*) > \phi(x_1).$$

From (2.6) and Lemma 2.3, we have

$$\frac{y^{*\top} F\left(x^*\right)}{y^{*\top} G\left(x^*\right)} \max_{\beta \in U} \left(\frac{\beta^\top F\left(x_1\right)}{\beta^\top G\left(x_1\right)}\right) \geq \frac{y^{*\top} F\left(x_1\right)}{y^{*\top} G\left(x_1\right)}.$$

It follows that

$$A(x_1) = {y^*}^\top G(x^*)\, {y^*}^\top F(x_1) - {y^*}^\top F(x^*)\, {y^*}^\top G(x_1) < 0 = A(x^*). \tag{3.1}$$

Thus, we have

$${y^*}^\top G(x^*)\left[{y^*}^\top F(x_1) - {y^*}^\top F(x^*)\right] - {y^*}^\top F(x^*)\left[{y^*}^\top G(x_1) - {y^*}^\top G(x^*)\right] < 0. \tag{3.2}$$

Using both the feasibility x_1 for (P) and the equality (2.5), we have

$$B(x_1) \leq 0 = B(x^*). \tag{3.3}$$

Consequently, expressions (3.1) and (3.3) yield

$$C(x_1) < C(x^*). \tag{3.4}$$

By (2.4), there exist $\xi \in \partial({y^*}^\top F)(x^*)$, $\zeta \in \partial({z^*}^\top H)(x^*)$, and $\mu \in \partial(-{y^*}^\top G)(x^*)$, such that

$${y^*}^\top G(x^*)(\xi + \zeta) + {y^*}^\top F(x^*)\,\mu = 0.$$

From here it results that

$$\Im\left(x_1, x^*; {y^*}^\top G(x^*)(\xi + \zeta) + {y^*}^\top F(x^*)\,\mu\right) = 0. \tag{3.5}$$

If Hypothesis (a) holds, the following inequalities are valid:

$${y^*}^\top F(x_1) - {y^*}^\top F(x^*) \geq \Im(x_1, x^*; \xi) + \rho_1\theta(x_1, x^*), \tag{3.6}$$

$$-\left[{y^*}^\top G(x_1) - {y^*}^\top G(x^*)\right] \geq \Im(x_1, x^*; \mu) + \rho_2\theta(x_1, x^*), \tag{3.7}$$

$${z^*}^\top H(x_1) - {z^*}^\top H(x^*) \geq \Im(x_1, x^*; \zeta) + \rho_3\theta(x_1, x^*). \tag{3.8}$$

Now, multiplying (3.6) by ${y^*}^\top G(x^*)$, (3.7) by ${y^*}^\top F(x^*)$, and (3.8) by ${y^*}^\top G(x^*)$, and adding the resulting inequalities and with the nonnegativity of ${y^*}^\top G(x^*)$ and ${y^*}^\top F(x^*)$, and (3.5), (3.2), (3.3), and the sublinearity of $\Im$, we have

$$0 > \left({y^*}^\top G(x^*)\,\rho_1 + {y^*}^\top F(x^*)\,\rho_2 + {y^*}^\top G(x^*)\,\rho_3\right)\theta(x_1, x^*),$$

which is a contradiction to the fact that

$${y^*}^\top G(x^*)\,\rho_1 + {y^*}^\top F(x^*)\,\rho_2 + {y^*}^\top G(x^*)\,\rho_3 \geq 0.$$

If Hypothesis (b) holds, using the $(\Im, \rho_1, \theta)$-pseudoconvexity of A at x^* and the inequality (3.1), we have

$$\Im\left(x_1, x^*; {y^*}^\top G(x^*)\,\xi + {y^*}^\top F(x^*)\,\mu\right) < -\rho_1\theta(x_1, x^*). \tag{3.9}$$

Using the $(\Im, \rho_2, \theta)$-quasiconvexity of B at x^*, we get from (3.3)

$$\Im(x_1, x^*; \zeta) \leq -\rho_2\theta(x_1, x^*). \tag{3.10}$$

From (3.9), (3.10), (3.5), and the nonnegativity of ${y^*}^\top G(x^*)$, and the sublinearity of $\Im$, we have

$$\left(\rho_1 + {y^*}^\top G(x^*)\,\rho_2\right)\theta(x_1, x^*) < 0,$$

which is a contradiction to the fact that

$$\rho_1 + {y^*}^\top G(x^*)\,\rho_2 \geq 0.$$

The proof of the theorem under the Hypotheses (c) and (d) can be carried out along with the same lines of (b). If Hypothesis (e) holds, using the $(\Im, \rho, \theta)$-pseudoconvexity of C at x^* and the inequality (3.4), we have

$$\Im\left(x_1, x^*; {y^*}^\top G(x^*)(\xi + \zeta) + {y^*}^\top F(x^*)\,\mu\right) < -\rho\theta(x_1, x^*). \tag{3.11}$$

Consequently, inequalities (3.11) and (3.5) yield

$$\rho\theta(x_1, x^*) < 0,$$

which is a contradiction to the fact that $\rho \geq 0$. Hypothesis (f) follows along with the same lines as (e). Hence, the proof is complete. ∎

4. THE FIRST DUAL MODEL—WOLFE TYPE DUAL

Utilizing Theorem 2.2, in Sections 4 and 5 we shall introduce two parametric-free dual models and prove appropriate duality theorems. Indeed, we shall demonstrate that the following is a dual problem for (P):

$$\begin{aligned} &\text{maximize} && \frac{\left(y^\top F(u) + z^\top H(u)\right)}{y^\top G(u)} \\ &\text{subject to} && 0 \in y^\top G(u)\left(\partial\left(y^\top F\right)(u) + \partial\left(z^\top H\right)(u)\right) \\ &&& - \left(y^\top F(u) + z^\top H(u)\right)\partial\left(y^\top G\right)(u), \qquad (4.1) \\ &&& y \in U, \quad z \in \mathbb{R}^m_+. \qquad (4.2) \end{aligned} \qquad \text{(DI)}$$

We denote by K_1 the set of all feasible solutions $(u, y, z) \in X_0 \times U \times \mathbb{R}^m_+$ of problem (DI). We assume throughout this section that $y^\top F(u) + z^\top H(u) \geq 0$ and $y^\top G(u) > 0$.

THEOREM 4.1. WEAK DUALITY. *Let $x \in S$ and $(u, y, z) \in K_1$ and assume that*

$$D(\cdot) = y^\top G(u)\left[y^\top F(\cdot) + z^\top H(\cdot)\right] - y^\top G(\cdot)\left[y^\top F(u) + z^\top H(u)\right].$$

If any one of the following conditions holds:

(a) $y^\top F$ *is* $(\Im, \rho_1, \theta)$*-convex,* $-y^\top G$ *is* $(\Im, \rho_2, \theta)$*-convex,* $z^\top H$ *is* $(\Im, \rho_3, \theta)$*-convex, and* $y^\top G(u)\rho_1 + [y^\top F(u) + z^\top H(u)]\rho_2 + y^\top G(u)\rho_3 \geq 0$;

(b) D *is* $(\Im, \rho, \theta)$*-pseudoconvex and* $\rho \geq 0$;

(c) D *is Ponstein* $(\Im, \rho, \theta)$*-quasiconvex and* $\rho > 0$;

then

$$\phi(x) \geq \frac{\left(y^\top F(u) + z^\top H(u)\right)}{y^\top G(u)}.$$

PROOF. Suppose

$$\phi(x) < \frac{\left(y^\top F(u) + z^\top H(u)\right)}{y^\top G(u)}.$$

Then, by Lemma 2.3 and $y \in U$, we have

$$\frac{y^\top F(x)}{y^\top G(x)} < \frac{\left(y^\top F(u) + z^\top H(u)\right)}{y^\top G(u)}.$$

It follows that

$$y^\top G(u) y^\top F(x) - y^\top G(x)\left[y^\top F(u) + z^\top H(u)\right] < 0.$$

Thus, we have

$$\begin{aligned} &\left[y^\top F(x) - y^\top F(u)\right] y^\top G(u) - \left[y^\top F(u) + z^\top H(u)\right]\left[y^\top G(x) - y^\top G(u)\right] - z^\top H(u) y^\top G(u) \\ &\quad = y^\top G(u) y^\top F(x) - y^\top G(x)\left[y^\top F(u) + z^\top H(u)\right] < 0, \end{aligned} \tag{4.3}$$

and another inequality

$$y^\top G(u)\left[y^\top F(x) + z^\top H(x)\right] - y^\top G(x)\left[y^\top F(u) + z^\top H(u)\right] < y^\top G(u) z^\top H(x).$$

Using the fact $y^\top G(u) > 0$, $z^\top H(x) \leq 0$, and the latest inequality, we have

$$D(x) < 0 = D(u). \tag{4.4}$$

By (4.1), there exist $\xi \in \partial(y^\top F)(u)$, $\zeta \in \partial(z^\top H)(u)$, and $\mu \in \partial(-y^\top G)(u)$, such that

$$y^\top G(u)(\xi + \zeta) + \left[y^\top F(u) + z^\top H(u)\right]\mu = 0.$$

From here it results that

$$\Im\left(x, u; y^\top G(u)(\xi + \zeta) + \left[y^\top F(u) + z^\top H(u)\right] \mu\right) = 0. \tag{4.5}$$

If Hypothesis (a) holds, the following inequalities are valid:

$$y^\top F(x) - y^\top F(u) \geq \Im(x, u; \xi) + \rho_1 \theta(x, u), \tag{4.6}$$

$$-\left[y^\top G(x) - y^\top G(u)\right] \geq \Im(x, u; \mu) + \rho_2 \theta(x, u), \tag{4.7}$$

$$\begin{aligned} -z^\top H(u) &\geq -z^\top H(x) + \Im(x, u; \zeta) + \rho_3 \theta(x, u) \\ &\geq \Im(x, u; \zeta) + \rho_3 \theta(x, u) \qquad (4.8) \\ &\text{(by (4.2) and the feasibility } x \text{ for (P)).} \end{aligned}$$

Now, multiplying (4.6) by $y^\top G(u)$, (4.7) by $y^\top F(u) + z^\top H(u)$, and (4.8) by $y^\top G(u)$, and adding the resulting inequalities and with the nonnegativity of $y^\top G(u)$ and $y^\top F(u) + z^\top H(u)$, and (4.3), (4.5), and the sublinearity of $\Im$, we have

$$0 > \left[y^\top G(u)\rho_1 + \left[y^\top F(u) + z^\top H(u)\right] \rho_2 + y^\top G(u)\rho_3\right] \theta(x, u),$$

which is a contradiction to the fact that

$$y^\top G(u)\rho_1 + \left[y^\top F(u) + z^\top H(u)\right] \rho_2 + y^\top G(u)\rho_3 \geq 0.$$

If Hypothesis (b) holds, using the $(\Im, \rho, \theta)$-pseudoconvexity of D and the inequality (4.4), we have

$$\Im\left(x, u; y^\top G(u)(\xi + \zeta) + \left[y^\top F(u) + z^\top H(u)\right] \mu\right) < -\rho\theta(x, u). \tag{4.9}$$

Consequently, inequalities (4.5) and (4.9) yield

$$\rho\theta(x, u) < 0,$$

which is a contradiction to the fact that $\rho \geq 0$. Hypothesis (c) follows along with the same lines as (b). Hence, the proof is complete. ∎

THEOREM 4.2. STRONG DUALITY. *If x^* is an optimal solution of (P) and the constraints of (P) satisfy Slater's constraint qualification [24], then there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$, such that (x^*, y^*, z^*) is a feasible solution of (DI). Furthermore, if the conditions of Theorem 4.1 hold for all feasible solutions of (DI), then (x^*, y^*, z^*) is an optimal solution of (DI) and the optimal values of (P) and (DI) are equal; that is,* min(P) = max(DI).

PROOF. By Theorem 2.2, there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$, such that (x^*, y^*, z^*) is a feasible solution of (DI). Furthermore,

$$\frac{\left(y^{*\top} F(x^*) + z^{*\top} H(x^*)\right)}{y^{*\top} G(x^*)} = \frac{y^{*\top} F(x^*)}{y^{*\top} G(x^*)} = \phi(x^*).$$

Thus, optimality of (x^*, y^*, z^*) for (DI) follows from Theorem 4.1. ∎

THEOREM 4.3. STRICT CONVERSE DUALITY. *Let x_1 and (x^*, y_0, z_0) be optimal solutions of (P) and (DI), respectively, and assume that the assumptions of Theorem 4.2 are fulfilled. If*

$$D(\cdot) = y_0^\top G(x^*) \left[y_0^\top F(\cdot) + z_0^\top H(\cdot)\right] - y_0^\top G(\cdot) \left[y_0^\top F(x^*) + z_0^\top H(x^*)\right]$$

is a strictly $(\Im, \rho, \theta)$-pseudoconvex and $\rho \geq 0$, then $x_1 = x^*$; that is, x^* is an optimal solution of (P) with the same optimal values $\phi(x_1) = (y_0^\top F(x^*) + z_0^\top H(x^*))/y_0^\top G(x^*)$.

PROOF. Suppose, on the contrary, that $x_1 \neq x^*$. From Theorem 4.2, we know that there exist $y_1 \in U$ and $z_1 \in \mathbb{R}_+^m$, such that (x_1, y_1, z_1) is an optimal solution of (DI) and

$$\phi(x_1) = \frac{\left(y_1^\top F(x_1) + z_1^\top H(x_1)\right)}{y_1^\top G(x_1)}.$$

Now proceeding as in the proof of Theorem 4.1 (replacing x by x_1 and (u, y, z) by (x^*, y_0, z_0)), we arrive at the following strict inequality:

$$\phi(x_1) > \frac{\left(y_0^\top F(x^*) + z_0^\top H(x^*)\right)}{y_0^\top G(x^*)}.$$

This contradicts the fact that

$$\phi(x_1) = \frac{\left(y_1^\top F(x_1) + z_1^\top H(x_1)\right)}{y_1^\top G(x_1)} = \frac{\left(y_0^\top F(x^*) + z_0^\top H(x^*)\right)}{y_0^\top G(x^*)}.$$

Therefore, we conclude that

$$x_1 = x^* \qquad \text{and} \qquad \phi(x_1) = \frac{\left(y_0^\top F(x^*) + z_0^\top H(x^*)\right)}{y_0^\top G(x^*)}. \qquad \blacksquare$$

5. SECOND DUAL MODEL—MOND-WEIR TYPE DUAL

We shall continue our discussion of parameter-free duality model for (P) in this section by showing that the following problem (DII) is also dual problem for (P):

$$\begin{aligned}
&\text{maximize} && \frac{y^\top F(u)}{y^\top G(u)} && \\
&\text{subject to} && 0 \in y^\top G(u) \left(\partial \left(y^\top F\right)(u) + \partial \left(z^\top H\right)(u)\right) - y^\top F(u) \partial \left(y^\top G\right)(u), && (5.1) \\
& && z^\top H(u) \geq 0, && (5.2) \\
& && y \in U, \qquad z \in \mathbb{R}_+^m. && (5.3)
\end{aligned} \qquad \text{(DII)}$$

We denote by K_2 the set of all feasible solutions $(u, y, z) \in X_0 \times U \times \mathbb{R}_+^m$ of problem (DII). Throughout this section, we assume that $y^\top F(u) \geq 0$ and $y^\top G(u) > 0$. Then, we can prove the following weak duality, strong duality, and strict converse duality theorems.

THEOREM 5.1. WEAK DUALITY. *Let $x \in S$ and $(u, y, z) \in K_2$ and let*

$$\begin{aligned}
E(\cdot) &= y^\top G(u) y^\top F(\cdot) - y^\top F(u) y^\top G(\cdot), \\
I(\cdot) &= z^\top H(\cdot), \qquad \text{and} \\
J(\cdot) &= E(\cdot) + y^\top G(u) I(\cdot).
\end{aligned}$$

If any one of the following conditions holds:

(a) *$y^\top F$ is $(\Im, \rho_1, \theta)$-convex, $-y^\top G$ is $(\Im, \rho_2, \theta)$-convex, $z^\top H$ is $(\Im, \rho_3, \theta)$-convex, and $y^\top G(u)\rho_1 + y^\top F(u)\rho_2 + y^\top G(u)\rho_3 \geq 0$;*
(b) *E is $(\Im, \rho_1, \theta)$-pseudoconvex, I is $(\Im, \rho_2, \theta)$- quasiconvex, and $\rho_1 + y^\top G(u)\rho_2 \geq 0$;*
(c) *E is $(\Im, \rho_1, \theta)$-quasiconvex, I is strictly $(\Im, \rho_2, \theta)$-pseudoconvex, and $\rho_1 + y^\top G(u)\rho_2 \geq 0$;*
(d) *E is Ponstein $(\Im, \rho_1, \theta)$-quasiconvex, I is $(\Im, \rho_2, \theta)$-quasiconvex, and $\rho_1 + y^\top G(u)\rho_2 > 0$;*
(e) *J is $(\Im, \rho, \theta)$-pseudoconvex and $\rho \geq 0$;*
(f) *J is Ponstein $(\Im, \rho, \theta)$-quasiconvex and $\rho > 0$;*

then

$$\phi(x) \geq \frac{y^\top F(u)}{y^\top G(u)}.$$

PROOF. By (5.1), there exist $\xi \in \partial(y^\top F)(u)$, $\zeta \in \partial(z^\top H)(u)$, and $\mu \in \partial(-y^\top G)(u)$, such that

$$y^\top G(u)(\xi + \zeta) + y^\top F(u)\mu = 0.$$

From here it results that

$$\Im\left(x, u; y^\top G(u)(\xi + \zeta) + y^\top F(u)\mu\right) = 0. \tag{5.4}$$

We suppose that

$$\phi(x) < \frac{y^\top F(u)}{y^\top G(u)}.$$

Then, by Lemma 2.3 and $y \in U$, we have

$$\frac{y^\top F(x)}{y^\top G(x)} < \frac{y^\top F(u)}{y^\top G(u)}.$$

It follows that

$$E(x) = y^\top G(u) y^\top F(x) - y^\top G(x) y^\top F(u) < 0 = E(u). \tag{5.5}$$

Thus, we have

$$y^\top G(u)\left[y^\top F(x) - y^\top F(u)\right] - y^\top F(u)\left[y^\top G(x) - y^\top G(u)\right] < 0. \tag{5.6}$$

Using both the feasibility x for (P) and the inequality (5.2), we have

$$I(x) \leq 0 \leq I(u). \tag{5.7}$$

Consequently, the inequalities (5.5) and (5.7) yield

$$J(x) < J(u). \tag{5.8}$$

If Hypothesis (a) holds, the following inequalities are valid:

$$y^\top F(x) - y^\top F(u) \geq \Im(x, u; \xi) + \rho_1 \theta(x, u), \tag{5.9}$$

$$-\left[y^\top G(x) - y^\top G(u)\right] \geq \Im(x, u; \mu) + \rho_2 \theta(x, u), \tag{5.10}$$

$$z^\top H(x) - z^\top H(u) \geq \Im(x, u; \zeta) + \rho_3 \theta(x, u). \tag{5.11}$$

Now, multiplying (5.9) by $y^\top G(u)$, (5.10) by $y^\top F(u)$, and (5.11) by $y^\top G(u)$, and adding the resulting inequalities and with the nonnegativity of $y^\top G(u)$ and $y^\top F(u)$, and (5.4), (5.6), (5.7), and the sublinearity of $\Im$, we have

$$0 > \left(y^\top G(u)\rho_1 + y^\top F(u)\rho_2 + y^\top G(u)\rho_3\right)\theta(x, u),$$

which is a contradiction to the fact that

$$y^\top G(u)\rho_1 + y^\top F(u)\rho_2 + y^\top G(u)\rho_3 \geq 0.$$

If Hypothesis (b) holds, using the $(\Im, \rho_1, \theta)$-pseudoconvexity of E and the inequality (5.5), we have

$$\Im\left(x, u; y^\top G(u)\xi + y^\top F(u)\mu\right) < -\rho_1 \theta(x, u). \tag{5.12}$$

Using the $(\Im, \rho_2, \theta)$-quasiconvexity of I, we get from (5.7)

$$\Im(x, u; \zeta) \leq -\rho_2 \theta(x, u). \tag{5.13}$$

From (5.12), (5.13), (5.4), and the nonnegativity of $y^\top G(u)$, and the sublinearity of $\Im$, we have

$$(\rho_1 + y^\top G(u)\rho_2)\, \theta(x, u) < 0,$$

which is a contradiction to the fact that

$$\rho_1 + y^\top G(u)\rho_2 \geq 0.$$

The proof of the theorem under the Hypotheses (c) and (d) can be carried out along with the same lines of (b). If Hypothesis (e) holds, using the $(\Im, \rho, \theta)$-pseudoconvexity of J and the inequality (5.8), we have

$$\Im\left(x, u; y^\top G(u)(\xi + \zeta) + y^\top F(u)\mu\right) < -\rho\theta(x, u). \tag{5.14}$$

Consequently, inequalities (5.14) and (5.4) yield

$$\rho\theta(x, u) < 0,$$

which is a contradiction to the fact that $\rho \geq 0$. Hypothesis (f) follows along with the same lines as (e). Hence, the proof is complete. ∎

THEOREM 5.2. STRONG DUALITY. *If x^* is an optimal solution of (P) and the constraints of (P) satisfy Slater's constraint qualification [24], then there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$, such that (x^*, y^*, z^*) is a feasible solution of (DII). Furthermore, if the conditions of Theorem 5.1 hold for all feasible solutions of (DII), then (x^*, y^*, z^*) is an optimal solution of (DII) and the optimal values of (P) and (DII) are equal; that is,* $\min(\mathrm{P}) = \max(\mathrm{DII})$.

PROOF. By Theorem 2.2, there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$ such that (x^*, y^*, z^*) is a feasible solution of (DII). Since (P) and (DII) have the same objective function, the optimality of (x^*, y^*, z^*) for (DII) follows from Theorem 5.1. ∎

THEOREM 5.3. STRICT CONVERSE DUALITY. *Let x_1 and (x^*, y_0, z_0) be optimal solutions of (P) and (DII), respectively, and assume that the assumptions of Theorem 5.2 are fulfilled. If $E(\cdot) = {y_0}^\top G(x^*) {y_0}^\top F(\cdot) - {y_0}^\top F(x^*) {y_0}^\top G(\cdot)$ is a strictly $(\Im, \rho_1, \theta)$-pseudoconvex and $I(\cdot) = {z_0}^\top H(\cdot)$ is a $(\Im, \rho_2, \theta)$-quasiconvex, and $\rho_1 + \rho_2 \geq 0$, then $x_1 = x^*$; that is, x^* is an optimal solution of (P) with the same optimal values $\phi(x_1) = y_0^\top F(x^*)/y_0^\top G(x^*)$.*

PROOF. Suppose on the contrary we assume that $x_1 \neq x^*$. From Theorem 5.2, we know that there exist $y_1 \in U$ and $z_1 \in \mathbb{R}^m_+$, such that (x_1, y_1, z_1) is an optimal solution of (DII) with optimal value

$$\phi(x_1) = \frac{y_1^\top F(x_1)}{y_1^\top G(x_1)}.$$

Now proceeding as in the proof of Theorem 5.1 (replacing x by x_1 and (u, y, z) by (x^*, y_0, z_0)), we arrive at the following strict inequality:

$$\phi(x_1) > \frac{y_0^\top F(x^*)}{y_0^\top G(x^*)},$$

which contradicts the fact of

$$\phi(x_1) = \frac{y_1^\top F(x_1)}{y_1^\top G(x_1)} = \frac{y_0^\top F(x^*)}{y_0^\top G(x^*)}.$$

Therefore, we conclude that

$$x_1 = x^* \qquad \text{and} \qquad \phi(x_1) = \frac{y_0^\top F(x^*)}{y_0^\top G(x^*)}.$$

This completes the proof of the theorem. ∎

6. THE THIRD DUAL MODEL

Making use of Theorem 2.1, in this section we can formulate the following parametric dual problem:

$$\begin{array}{lll} \text{maximize} & v & \\ \text{subject to} & 0 \in \partial\left(y^\top F\right)(u) - v\partial\left(y^\top G\right)(u) + \partial\left(z^\top H\right)(u), & (6.1) \\ & y^\top F(u) - vy^\top G(u) \geq 0, & (6.2) \quad \text{(DIII)} \\ & z^\top H(u) \geq 0, & (6.3) \\ & y \in U, \quad v \in \mathbb{R}_+, \quad z \in \mathbb{R}_+^m. & (6.4) \end{array}$$

We denote by K_3 the set of all feasible solutions $(u, y, z, v) \in X_0 \times U \times \mathbb{R}_+^m \times \mathbb{R}_+$ of problem (DIII). Then by the similar proof as Theorems 5.1–5.3, we can obtain the following Theorems 6.1–6.3 relating (P) and (DIII).

THEOREM 6.1. WEAK DUALITY. *Let $x \in S$ and $(u, y, z, v) \in K_3$, and let*

$$\begin{aligned} L(\cdot) &= y^\top F(\cdot) - vy^\top G(\cdot), \\ I(\cdot) &= z^\top H(\cdot), \qquad \text{and} \\ M(\cdot) &= L(\cdot) + I(\cdot). \end{aligned}$$

If any one of the following conditions holds:

(a) *$y^\top F$ is $(\Im, \rho_1, \theta)$-convex, $-y^\top G$ is $(\Im, \rho_2, \theta)$-convex, $z^\top H$ is $(\Im, \rho_3, \theta)$-convex, and $\rho_1 + v\rho_2 + \rho_3 \geq 0$;*
(b) *L is $(\Im, \rho_1, \theta)$-pseudoconvex, I is $(\Im, \rho_2, \theta)$-quasiconvex, and $\rho_1 + \rho_2 \geq 0$;*
(c) *L is $(\Im, \rho_1, \theta)$-quasiconvex, I is strictly $(\Im, \rho_2, \theta)$-pseudoconvex, and $\rho_1 + \rho_2 \geq 0$;*
(d) *E is Ponstein $(\Im, \rho_1, \theta)$-quasiconvex, I is $(\Im, \rho_2, \theta)$-quasiconvex, and $\rho_1 + \rho_2 > 0$;*
(e) *M is $(\Im, \rho, \theta)$-pseudoconvex and $\rho \geq 0$;*
(f) *M is Ponstein $(\Im, \rho, \theta)$-quasiconvex and $\rho > 0$;*

then

$$\phi(x) \geq v.$$

THEOREM 6.2. STRONG DUALITY. *If x^* is an optimal solution of (P) and the constraints of (P) satisfy Slater's constraint qualification [24], then there exist $y^* \in U$, $z^* \in \mathbb{R}_+^m$ and $v^* \in \mathbb{R}_+$, such that (x^*, y^*, z^*, v^*) is a feasible solution of (DIII). Furthermore, if the conditions of Theorem 6.1 hold for all feasible solutions of (DIII), then (x^*, y^*, z^*, v^*) is an optimal solution of (DIII) and the optimal values of (P) and (DIII) are equal; that is,* $\min(\mathrm{P}) = \max(\mathrm{DIII})$.

THEOREM 6.3. STRICT CONVERSE DUALITY. *Let x_1 and (x^*, y_0, z_0, v_0) be optimal solutions of (P) and (DIII), respectively, and assume that the assumptions of Theorem 6.2 are fulfilled. If ${y_0}^\top F(\cdot) - v_0 {y_0}^\top G(\cdot)$ is a strictly $(\Im, \rho_1, \theta)$-pseudoconvex and $I(\cdot) = {z_0}^\top H(\cdot)$ is a $(\Im, \rho_2, \theta)$-quasiconvex, and $\rho_1 + \rho_2 \geq 0$, then $x_1 = x^*$; that is, x^* is an optimal solution of (P) with the same optimal values $\phi(x_1) = v_0$.*

REFERENCES

1. H.C. Lai, J.C. Liu, and K. Tanaka, Duality without a constraint qualification for generalized fractional programming, *J. Optim. Theory Appl.* (to appear).
2. C.R. Bector and S.K. Suneja, Duality in nondifferentiable generalized fractional programming, *Asia-Pacific J. Opera. Re.* **5**, 134–139, (1988).
3. C.R. Bector, S. Chandra and M.K. Bector, Generalized fractional programming duality: A parametric approach, *J. Optim. Theory. Appl.* **60**, 243–260, (1989).
4. C.R. Bector, S. Chandra and V. Kumar, Duality for minimax programming involving V-invex functions, *Optimization* **30**, 93–103, (1994).

5. S. Chandra, B.D. Craven and B. Mond, Generalized fractional programming duality: A ratio game approach, *J. Aust. Math. Soc. Series B* **28**, 170–180, (1986).
6. J.P. Crouzeix, J.A. Ferland and S. Schaible, Duality in generalized fractional programming, *Math. Prog.* **27**, 342–354, (1983).
7. J.P. Crouzeix, J.A. Ferland and S. Schaible, An algorithm for generalized fractional programs, *J. Optim. Theory. Appl.* **47**, 35–49, (1985).
8. H.C. Lai and J.C. Liu, Duality for a minimax programming problem containing n-set functions, *J. Math. Anal. Appl.* (to appear).
9. H.C. Lai and J.C. Liu, On minimax fractional programming of generalized convex set functions, Preprint.
10. J.C. Liu, Optimality and duality for generalized fractional programming involving nonsmooth pseudoinvex functions, *J. Math. Anal. Appl.* **202**, 667–685, (1996).
11. J.C. Liu, Optimality and duality for generalized fractional programming involving nonsmooth (F, ρ)-convex functions, *Computers Math. Applic.* **32** (2), 91–102, (1996).
12. J.C. Liu, Optimality and duality for generalized fractional variational problems involving generalized (F, ρ)-convex functions, *Optimization* **37**, 369–383, (1996).
13. J.C. Liu, Complex minimax programming, *Util. Math.* **37**, 369–383, (1996).
14. J.C. Liu, C.S. Wu and R.L. Sheu, Duality for fractional minimax programming, *Optimization* **41**, 117–133, (1997).
15. J.C. Liu and C.S. Wu, On minimax fractional optimality conditions with invexity, *J. Math. Anal. Appl.* **219**, 21–35, (1998).
16. J.C. Liu and C.S. Wu, On minimax fractional optimality conditions with (F, ρ)-convexity, *J. Math. Anal. Appl.* **219**, 36–51, (1998).
17. V. Preda, On minimax programming problems containing n-set functions, *Optimization* **22**, 527–537, (1991).
18. G.J. Zalmai, Optimality conditions and duality models for generalized fractional programming problems containing locally subdifferentiable and ρ-convex functions, *Optimization* **32**, 95–124, (1995).
19. G.J. Zalmai, Optimality conditions and duality for a class of continuous-time generalized fractional programming problems, *J. Math. Anal. Appl.* **153**, 356–371, (1990).
20. G.J. Zalmai, Duality for generalized fractional programs involving n-set functions, *J. Math. Anal. Appl.* **149**, 339–350, (1990).
21. G.J. Zalmai, Optimality conditions and duality for constrained measurable subset selection problems with minimax objective functions, *Optimization* **20**, 377–395, (1989).
22. A. Ben-Israel, A. Ben-Tal and S. Zlobec, *Optimality in Nonlinear Programming: A Feasible Direction Approach*, John Wiley and Sons, New York, (1981).
23. H.F. Clarke, *Optimization and Nonsmooth Analysis*, Wiley-Interscience, New York, (1983).
24. J.B. Hiriart-Urruty, On optimality conditions in nondifferentiable programming, *Math. Prog.* **14**, 73–86, (1978).
25. J. Ponstein, Seven kinds of convexity, *SIAM Rev.* **9**, 115–119, (1969).
26. D. Bhatia and P. Jain, Generalized (F, ρ)-convexity and duality for nonsmooth multi-objective programs, *Optimization* **31**, 153–164, (1994).
27. H.C. Lai and J.C. Liu, Optimality conditions for multiobjective programming with generalized $(\Im, \rho, \theta)$-convex set functions, *J. Math. Anal. Appl.* **215**, 443–460, (1997).
28. J.C. Liu, Duality for nondifferentiable static multiobjective variational problems involving generalized (F, ρ) functions, *Computers Math. Applic.* **31** (12), 77–89, (1996).
29. J.C. Liu, Optimality and duality for multiobjective fractional programming involving nonsmooth (F, ρ) functions, *Optimization* **36**, 333–346, (1996).
30. J.C. Liu, Sufficient optimality criteria and duality for nondifferentiable fractional variational problems with generalized (F, ρ)-convexity, *Optimization* (to appear).
31. V. Preda, On efficiency and duality for multiobjective programs, *J. Math. Anal. Appl.* **166**, 365–377, (1992).
32. V. Preda, On sufficiency and duality for generalized quasiconvex programs, *J. Math. Anal. Appl.* **181**, 77–88, (1994).
33. G.J. Zalmai, Optimality conditions and duality models for a class of nonsmooth constrained fractional variational problems, *Optimization* **30**, 15–51, (1994).